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On the Thermal Stresses in Beams

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SUMMARY

The present paper is concerned with the problem of thermal stresses in beams when the temperature distribution is a polynomial in the axial coordinate where the coefficients of the polynomial are functions of the two remaining coordinates. We consider the case of the homogeneous and isotropic beams and the case of the composite beams—when the outer contour of the cross-section contains an arbitrary number of other contours each enclosing a different homogeneous isotropic material.

1. Introduction

The general theory of the thermal stresses in elastic beams has been studied only when the temperature distribution is restricted to a polynomial of the first degree in the axial coordinate (see e.g. [1]).

In this paper we consider the mathematical problem of the thermal stresses in beams when the temperature distribution is a polynomial of the r degree in the axial coordinate x_3, namely

$$T = \sum_{k=0}^{r} T_k(x_1, x_2) x_3^k . \tag{1.1}$$

We assume that the functions $T_k(x_1, x_2)$ are given.

We consider homogeneous beams and composite beams. In the later case we assume that the outer contour of the cross-section contains an arbitrary number of other contours each enclosing a different homogeneous isotropic material, i.e. a cylindrical beam with longitudinal holes which are completely filled with beams of different homogeneous isotropic materials. We shall assume the materials to be welded together along the interfaces.

2. Basic Equations

We consider a cylindrical beam of length l bounded by plane ends perpendicular to the generators. The cross-section Σ is assumed to be a simply-connected region, bounded by a closed Liapunov curve L. We suppose that body force is absent and the lateral surface is free of applied force. We suppose the beam to be fixed at one end and to be kept in thermoelastic equilibrium under the action of a given temperature, the loading applied to the free end being statically equivalent to zero.

We take the right-hand axes of reference Ox_1, Ox_2 in the plane of the free end and Ox_3 directed parallel to the generators and into the material.

The basic equations in the linear static theory of thermoelasticity for homogeneous and isotropic solids are [1]:

– equilibrium equations

$$t_{ij,j} = 0 , \tag{2.1}$$

– constitutive equations

$$t_{ij} = \frac{E\nu}{(1+\nu)(1-2\nu)} e_{rr} \delta_{ij} + \frac{E}{1+\nu} e_{ij} - \frac{E\alpha}{1-2\nu} T \delta_{ij} , \tag{2.2}$$

– strain-displacement relations

$$2e_{ij} = u_{i,j} + u_{j,i} . \tag{2.3}$$

In these equations we have used the following notations: t_{ij} are components of the stress tensor, e_{ij} are components of the strain tensor, T is the temperature measured from the constant absolute temperature of the natural state, u_i are components of displacement vector, δ_{ij} is the Kronecker's delta, E, ν, α are the characteristic constants of the material, and the comma denotes partial derivation with respect to the variables x_i.

On the lateral surface of the beam we have the following conditions

$$t_{i\alpha} n_\alpha = 0\,, \qquad (i = 1, 2, 3\ ;\ \alpha = 1, 2)\,, \tag{2.4}$$

where $(n_1, n_2, 0)$ are the direction cosines of the outward normal to the lateral surface.

On the plane $x_3 = 0$ we have the following conditions [2]

$$\int_\Sigma t_{13} d\sigma = 0\,, \qquad \int_\Sigma t_{23} d\sigma = 0\,, \tag{2.5}$$

$$\int_\Sigma t_{33} d\sigma = 0\,, \tag{2.6}$$

$$\int_\Sigma x_2 t_{33} d\sigma = \int_\Sigma x_1 t_{33} d\sigma = 0\,, \tag{2.7}$$

$$\int_\Sigma (x_1 t_{23} - x_2 t_{13}) d\sigma = 0\,. \tag{2.8}$$

We assumed that the temperature distribution has the form (1.1). Let us denote by (A) the problem of determination of thermal stresses in the considered beam when the temperature distribution has the form

$$T = f(x_1, x_2)\, x_3^n\,, \tag{2.9}$$

where n is a positive integer or zero, and the function $f(x_1, x_2)$ is known.

Obviously, if we know the solving of the problem (A) for any n then, according to the linearity of the problem, we can determine the solution in the case (1.1).

We denote by (B) the problem of determination of thermal stresses in the same beam if the temperature has the form

$$T = f(x_1, x_2)\, x_3^{n+1}\,, \tag{2.10}$$

and the problem (A) is assumed to be solved.

If the problem (B) is solved and we know the solution of the problem (A) for $n=0$, then we can obtain the solution for $n=1$, and so on. This fact leads to the solution of the problem in which the temperature has the form (1.1).

Thus, to solve the initial problem we must solve the problem (A) for $n=0$ and the problem (B).

3. Homogeneous Beams

Let us consider a homogeneous and isotropic beam under the action of the temperature distribution $T = f(x_1, x_2)$.

We assume that the components of the displacement vector have the form

$$\begin{aligned} u_1 &= -\tfrac{1}{2} a [x_3^2 + \nu(x_1^2 - x_2^2)] - \nu x_1 (b x_2 + c) + v_1(x_1, x_2)\,, \\ u_2 &= -\tfrac{1}{2} b [x_3^2 - \nu(x_1^2 - x_2^2)] - \nu x_2 (a x_1 + c) + v_2(x_1, x_2)\,, \\ u_3 &= (a x_1 + b x_2 + c) x_3\,, \end{aligned} \tag{3.1}$$

where the functions $v_\alpha(x_1, x_2)$, $(\alpha = 1, 2)$ and the constants a, b, c will be determined in the following.

If we introduce the notations

$$2\varepsilon_{\beta\gamma}=v_{\beta,\gamma}+v_{\gamma,\beta},$$

$$\sigma_{\beta\gamma}=\frac{E\nu}{(1+\nu)(1-2\nu)}\varepsilon_{\rho\rho}\delta_{\beta\gamma}+\frac{E}{1+\nu}\varepsilon_{\beta\gamma}-\frac{E\alpha}{1-2\nu}T\delta_{\beta\gamma}, \qquad (\beta,\gamma,\rho=1,2), \tag{3.2}$$

from (2.2), (2.3), (3.1), (3.2) we obtain

$$t_{\beta\gamma}=\sigma_{\beta\gamma},\quad t_{\beta 3}=0,$$
$$t_{33}=E(ax_1+bx_2+c)+\nu\sigma_{\beta\beta}-E\alpha T, \qquad (\beta,\gamma=1,2) \tag{3.3}$$

The equilibrium equations (2.1) become

$$\sigma_{\alpha\beta,\beta}=0, \tag{3.4}$$

and the boundary conditions (2.4) are satisfied if

$$\sigma_{\alpha\beta}n_\beta=0,\ \text{on } L, \qquad (\alpha,\beta=1,2). \tag{3.5}$$

From (3.2), (3.4), (3.5) it follows that the functions v_α, $\varepsilon_{\alpha\beta}$, $\sigma_{\alpha\beta}$ satisfy the equations of the thermoelastic plane strain [1] for temperature distribution $T=f(x_1, x_2)$.

The conditions (2.5), (2.8) are satisfied on the basis of the relations (3.3). If the above thermoelastic plane strain problem is solved, from (2.6) and (2.7) we obtain

$$a=\frac{1}{Ed}\left[I_{11}M_2-I_{12}M_1+(x_2^0I_{12}-x_1^0I_{11})P\right],$$

$$b=\frac{1}{Ed}\left[I_{22}M_1-I_{12}M_2+(x_1^0I_{12}-x_2^0I_{22})P\right], \tag{3.6}$$

$$c=\frac{1}{ES}P-ax_1^0-bx_2^0,$$

where

$$P=\int_\Sigma F\,d\sigma,\quad M_1=\int_\Sigma x_2F\,d\sigma,\quad M_2=\int_\Sigma x_1F\,d\sigma,\quad F=E\alpha T-\nu\sigma_{\beta\beta},\quad S=\int_\Sigma d\sigma,$$

$$I_{11}=\int_\Sigma (x_2-x_2^0)^2d\sigma,\quad I_{12}=\int_\Sigma (x_1-x_1^0)(x_2-x_2^0)d\sigma,\quad I_{22}=\int_\Sigma (x_1-x_1^0)^2d\sigma,$$

$$d=I_{11}I_{22}-I_{12}^2, \tag{3.7}$$

and x_1^0, x_2^0 are the coordinates of the centroid of the Σ. Thus, the problem (A) for $n=0$ is reduced to a two-dimensional problem.

In what follows we seek to solve the problem (B). We denote by u_i^*, e_{ij}^*, t_{ij}^* $(i,j=1,2,3)$ respectively the components of displacement vector, the components of strain tensor, the components of stress tensor from the problem (A) and by u_i, e_{ij}, t_{ij} the analogous functions from the problem (B).

We assume that the functions u_i^*, e_{ij}^*, t_{ij}^* are known.

We try to solve the problem (B) assuming that the components of the displacement vector have the form

$$u_1=(n+1)\left[\int_0^{x_3}u_1^*dx_3-\nu x_1(\tfrac{1}{2}ax_1+bx_2+c)-\tau x_2x_3-\tfrac{1}{2}ax_3^2+\tfrac{1}{2}a\nu x_2^2+v_1(x_1,x_2)\right],$$

$$u_2=(n+1)\left[\int_0^{x_3}u_2^*dx_3-\nu x_2(ax_1+\tfrac{1}{2}bx_2+c)+\tau x_1x_3-\tfrac{1}{2}bx_3^2+\tfrac{1}{2}b\nu x_1^2+v_2(x_1,x_2)\right],$$

$$u_3=(n+1)\left[\int_0^{x_3}u_3^*dx_3+x_3(ax_1+bx_2+c)+\Phi(x_1,x_2)\right], \tag{3.8}$$

where the functions $v_\alpha(x_1,x_2)$, $(\alpha=1,2)$, $\Phi(x_1,x_2)$ and the constants a, b, c, τ will be determined in the following.

From (2.3) and (3.8) we obtain

$$e_{\alpha\beta} = (n+1)\left[\int_0^{x_3} e^*_{\alpha\beta}\,dx_3 - \nu(ax_1+bx_2+c)\delta_{\alpha\beta}+\varepsilon_{\alpha\beta}\right],$$

$$e_{13} = (n+1)\left\{\int_0^{x_3} e^*_{13}\,dx_3 + \tfrac{1}{2}[\Phi_{,1}-\tau x_2+u^*_1(x_1,x_2,0)]\right\},$$

$$e_{23} = (n+1)\left\{\int_0^{x_3} e^*_{23}\,dx_3 + \tfrac{1}{2}[\Phi_{,2}+\tau x_1+u^*_2(x_1,x_2,0)]\right\},$$

$$e_{33} = (n+1)\left[\int_0^{x_3} e^*_{33}\,dx_3 + ax_1+bx_2+c+u^*_3(x_1,x_2,0)\right], \tag{3.9}$$

where

$$2\varepsilon_{\alpha\beta} = v_{\alpha,\beta}+v_{\beta,\alpha}, \qquad (\alpha,\beta=1,2). \tag{3.10}$$

If we note

$$\sigma_{\alpha\beta} = \frac{E\nu}{(1+\nu)(1-2\nu)}\varepsilon_{\rho\rho}\delta_{\alpha\beta} + \frac{E}{1+\nu}\varepsilon_{\alpha\beta}, \qquad (\alpha,\beta,\rho=1,2), \tag{3.11}$$

from (2.2), (2.9), (2.10), (3.9), (3.10) we get

$$t_{\alpha\beta} = (n+1)\left[\int_0^{x_3} t^*_{\alpha\beta}\,dx_3 + \sigma_{\alpha\beta}+\delta_{\alpha\beta}\lambda u^*_3(x_1,x_2,0)\right],$$

$$t_{13} = (n+1)\left\{\int_0^{x_3} t^*_{13}\,dx_3 + \mu[\Phi_{,1}-\tau x_2+u^*_1(x_1,x_2,0)]\right\},$$

$$t_{23} = (n+1)\left\{\int_0^{x_3} t^*_{23}\,dx_3 + \mu[\Phi_{,2}+\tau x_1+u^*_2(x_1,x_2,0)]\right\},$$

$$t_{33} = (n+1)\left[\int_0^{x_3} t^*_{33}\,dx_3 + E(ax_1+bx_2+c)+\nu\sigma_{\alpha\alpha}+(\lambda+2\mu)u^*_3(x_1,x_2,0)\right], \tag{3.12}$$

where

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \quad 2\mu = \frac{E}{1+\nu}. \tag{3.13}$$

The first two of the equilibrium equations (2.1) give

$$\sigma_{\alpha\beta,\beta} = F_\alpha, \tag{3.14}$$

where

$$F_\alpha = -t^*_{\alpha 3}(x_1,x_2,0) - \lambda u^*_{3,\alpha}(x_1,x_2,0), \qquad (\alpha,\beta=1,2). \tag{3.15}$$

From the first two of the relations (2.4) we obtain the boundary conditions

$$\sigma_{\alpha\beta}n_\beta = -\lambda n_\alpha u^*_3(x_1,x_2,0), \text{ on } L. \tag{3.16}$$

We see that the functions $v_\alpha(x_1,x_2)$, $\varepsilon_{\alpha\beta}(x_1,x_2)$, $\sigma_{\alpha\beta}(x_1,x_2)$, $(\alpha,\beta=1,2)$ satisfy the equations of elastic plane strain [3] problem (3.10), (3.11), (3.14), (3.16).

Let us show the existence of the solution of this problem. We denote by $\tau_{\alpha\beta}$ a particular solution of the system (3.14) and write

$$\sigma_{\alpha\beta} = \tau_{\alpha\beta}+\sigma^0_{\alpha\beta}.$$

The functions $\sigma^0_{\alpha\beta}$ satisfy the system

$$\sigma^0_{\alpha\beta,\beta} = 0, \tag{3.17}$$

and the boundary conditions

$$\sigma^0_{\alpha\beta}n_\beta = f_\alpha, \text{ on } L, \qquad (\alpha,\beta=1,2), \tag{3.18}$$

where

$$f_\alpha = -\lambda n_\alpha u_3^*(x_1, x_2, 0) - \tau_{\alpha\beta} n_\beta . \tag{3.19}$$

The conditions for the existence of the solution of the boundary value problem (3.17), (3.18) are [3]

$$\int_L f_1 ds = 0 , \quad \int_L f_2 ds = 0 , \quad \int_L (x_1 f_2 - x_2 f_1) ds = 0 . \tag{3.20}$$

Using (3.15), (3.19) and the divergence theorem, we obtain

$$\int_L f_\alpha ds = -\int_\Sigma [\tau_{\alpha\beta,\beta} + \lambda u^*_{3,\alpha}(x_1, x_2, 0)] d\sigma =$$

$$-\int_\Sigma [F_\alpha + \lambda u^*_{3,\alpha}(x_1, x_2, 0)] d\sigma = \int_\Sigma t^*_{\alpha 3} dx_3 ,$$

$$\int_\Sigma (x_1 f_2 - x_2 f_1) d\sigma = \int_\Sigma [x_2 \tau_{1\beta,\beta} + x_2 \lambda u^*_{3,1}(x_1, x_2, 0) - x_1 \tau_{2\beta,\beta} - x_1 \lambda u^*_{3,2}(x_1, x_2, 0)] d\sigma =$$

$$= \int_\Sigma [x_1 t^*_{23}(x_1, x_2, 0) - x_2 t^*_{13}(x_1, x_2, 0)] d\sigma , \qquad (\alpha = 1, 2) ,$$

so that the conditions (3.20) are satisfied, because the functions t^*_{ij} satisfy the conditions (2.5), (2.8).

The components of stress tensor t_{23}, t_{13}, t_{33} given by (3.12) must satisfy the last of equilibrium equation. We obtain for unknown function $\Phi(x_1, x_2)$ the following equation

$$\Delta\Phi = -u^*_{\alpha,\alpha}(x_1, x_2, 0) - \frac{1}{\mu} t^*_{33}(x_1, x_2, 0) , \qquad (\alpha = 1, 2) , \tag{3.21}$$

where Δ is the two-dimensional Laplacian operator.

From the last of the boundary conditions (2.4) we obtain

$$\frac{\partial\Phi}{\partial n} = \tau(x_2 n_1 - x_1 n_2) - n_\alpha u^*_\alpha(x_1, x_2, 0) , \text{ on } L . \tag{3.22}$$

Let us show that the boundary value problem (3.21), (3.22) has a solution. Let φ be the solution of the equation

$$\Delta\varphi = 0 , \tag{3.23}$$

with the boundary condition

$$\frac{\partial\varphi}{\partial n} = x_2 n_1 - x_1 n_2 , \text{ on } L . \tag{3.24}$$

Obviously, the function φ exists. Let us introduce the function ψ by

$$\Phi = \tau\varphi + \chi + \psi , \tag{3.25}$$

where χ is a particular solution of the equation (3.21).

From (3.21)–(3.25) it follows that the function ψ satisfies the equation

$$\Delta\psi = 0 , \tag{3.26}$$

and the boundary condition

$$\frac{\partial\psi}{\partial n} = -u^*_\alpha(x_1, x_2, 0) n_\alpha - \frac{\partial\chi}{\partial n} \equiv f , \text{ on } L . \tag{3.27}$$

The condition for the existence of the function ψ is

$$\int_L f ds = 0 . \tag{3.28}$$

Using the divergence theorem we see that this condition is satisfied in our case

$$\int_L f ds = - \int_\Sigma [u^*_{\alpha,\alpha}(x_1, x_2, 0) + \Delta\chi] d\sigma = \frac{1}{\mu} \int_\Sigma t^*_{33}(x_1, x_2, 0) d\sigma = 0 ,$$

because the functions t^*_{ij} satisfy the conditions (2.5)–(2.8).

From (2.6) and (2.7) we obtain

$$a = \frac{1}{Ed} [I_{11} M_2 - I_{12} M_1 + (x_2^0 I_{12} - x_1^0 I_{11}) P] ,$$

$$b = \frac{1}{Ed} [I_{22} M_1 - I_{12} M_2 + (x_1^0 I_{12} - x_2^0 I_{22}) P] , \tag{3.29}$$

$$c = \frac{1}{ES} P - a x_1^0 - b x_2^0 ,$$

where

$$P = \int_\Sigma F d\sigma , \quad M_1 = \int_\Sigma x_2 F d\sigma , \quad M_2 = \int_\Sigma x_1 F d\sigma ,$$

$$F = - \nu \sigma_{\alpha\alpha} - (\lambda + 2\mu) u_3^* (x_1, x_2, 0) ,$$

$I_{\alpha\beta}$, S and d being given by (3.7).

From (2.8), (3.12), (3.25) we determine the constant τ

$$\tau D = \int_\Sigma \{ x_2 [\chi_{,1} + \psi_{,1} + u_1^*(x_1, x_2, 0)] - x_1 [\chi_{,2} + \psi_{,2} + u_2^*(x_1, x_2, 0)] \} d\sigma , \tag{3.30}$$

where D is the torsional rigidity [3]

$$D = \int_\Sigma (x_1^2 + x_2^2 + x_1 \varphi_{,2} - x_2 \varphi_{,1}) d\sigma . \tag{3.31}$$

It is known [3] that $D > 0$.

The conditions (2.5) are identically satisfied on the basis of the equilibrium equations and the boundary conditions. For example, for the first of them we have

$$\int_\Sigma t_{13} d\sigma = \int_\Sigma [t_{13} + x_1 (t_{13,1} + t_{23,2} + t_{33,3})] d\sigma = \int_L x_1 (t_{13} n_1 + t_{23} n_2) ds + + (n+1) \int_\Sigma x_1 t^*_{33} d\sigma = 0 , \quad x_3 = 0 . \tag{3.32}$$

4. Composite Beams

Let us assume that the cross-section Σ consists of the assembly of the regions Σ_0 and Σ_j $(j=1, 2, \ldots, m+1)$, Σ_0 being a multiply-connected region, bounded by the closed curves L_j $(j=1, 2, \ldots, m+1)$ possessing no common points; L_{m+1} is the outer boundary of the region Σ and contains the curves L_j $(j=1, 2, \ldots, m)$. All the Σ_j are finite and simply-connected, bounded by the corresponding curves L_j $(j=1, 2, \ldots, m)$. Let us assume that the matter filling each of the regions Σ_0 and Σ_j $(j=1, 2, \ldots, m)$ is homogeneous and isotropic, while passing from one medium to another the thermoelastic properties are different.

The displacement vector and the stress vector must be continuous in passing from one medium to another so that we have the conditions

$$[u_k]_i = [u_k]_0 , \tag{4.1}$$

$$[t_{k\alpha} n_\alpha]_i = [t_{k\alpha} n_\alpha]_0 , \text{ on } L_i , \tag{4.2}$$

$$t_{k\alpha} n_\alpha = 0 , \text{ on } L_{m+1} , \qquad (i = 1, 2, \ldots, m \; ; \; k = 1, 2, 3 \; ; \; \alpha = 1, 2) , \tag{4.3}$$

where we have indicated that the expressions from parentheses are calculated for the material from the regions Σ_i ($i=1, 2, \ldots, m$) and Σ_0. Let ν_i, E_i and α_i be the caracteristic constants of the material from the region Σ_i ($i=0, 1, 2, \ldots, m$).

We consider the three problems $P^{(k)}$ ($k=1, 2, 3$) of elastic ($T=0$) plane strain [3] without body forces, in which the components of the displacement vector $w_\alpha^{(k)}$ and the components of the stress tensor $\tau_{\alpha\beta}^{(k)}$ ($\alpha, \beta=1, 2$; $k=1, 2, 3$) satisfy the conditions

$$\begin{aligned}&[\tau_{\alpha\beta}^{(k)} n_\beta]_i = [\tau_{\alpha\beta}^{(k)} n_\beta]_0 ,\\ &[w_\alpha^{(k)}]_i - [w_\alpha^{(k)}]_0 = g_{i\alpha}^{(k)} , \text{ on } L_i ,\\ &\tau_{\alpha\beta}^{(k)} n_\beta = 0 , \text{ on } L_{m+1} , \qquad (i=1, 2, \ldots, m \; ; \; \alpha, \beta = 1, 2 \; ; \; k=1, 2, 3) ,\end{aligned} \tag{4.4}$$

where

$$\begin{aligned}&g_{i1}^{(1)} = \tfrac{1}{2}(\nu_i - \nu_0)(x_1^2 - x_2^2) , \quad g_{i2}^{(1)} = (\nu_i - \nu_0) x_1 x_2 ,\\ &g_{i1}^{(2)} = (\nu_i - \nu_0) x_1 x_2 , \qquad g_{i2}^{(2)} = -\tfrac{1}{2}(\nu_i - \nu_0)(x_1^2 - x_2^2) ,\\ &g_{i1}^{(3)} = (\nu_i - \nu_0) x_1 , \qquad g_{i2}^{(3)} = (\nu_i - \nu_0) x_2 .\end{aligned} \tag{4.5}$$

We assume that $w_\alpha^{(k)}$ and $\tau_{\alpha\beta}^{(k)}$ are known [3], [4].

Let us consider the following functions defined on Σ_i

$$\begin{aligned}w_1 &= -\tfrac{1}{2} a [x_3^2 + \nu_i (x_1^2 - x_2^2)] - b\nu_i x_1 x_2 - c\nu_i x_1 + v_1 (x_1, x_2) ,\\ w_2 &= -\tfrac{1}{2} b [x_3^2 - \nu_i (x_1^2 - x_2^2)] - a\nu_i x_1 x_2 - c\nu_i x_2 + v_2 (x_1, x_2) ,\\ w_3 &= (a x_1 + b x_2 + c) x_3 ,\end{aligned} \tag{4.6}$$

where the functions $v_\alpha(x_1, x_2)$, ($\alpha=1, 2$) and the constants a, b, c will be determined in the following.

We try to determine the solution of the stated problem assuming that the components of the displacements vector have the form

$$\begin{aligned}u_\alpha &= w_\alpha + a w_\alpha^{(1)} + b w_\alpha^{(2)} + c w_\alpha^{(3)} , \qquad (\alpha = 1, 2) ,\\ u_3 &= w_3 .\end{aligned} \tag{4.7}$$

Taking in account the relations (4.4), (4.5) we see that the functions (4.7) are continuous in passing from one medium to another if

$$[v_\alpha]_i = [v_\alpha]_0 , \text{ on } L_i \qquad (i=1, 2, \ldots, m \; ; \; \alpha = 1, 2) . \tag{4.8}$$

If we introduce the notations (3.2), from (2.2), (2.3), (4.7) we obtain

$$\begin{aligned}t_{\alpha\beta} &= \sigma_{\alpha\beta} + a\tau_{\alpha\beta}^{(1)} + b\tau_{\alpha\beta}^{(2)} + c\tau_{\alpha\beta}^{(3)} , \; t_{\alpha 3} = 0 , \qquad (\alpha, \beta = 1, 2) ,\\ t_{33} &= E_i (a x_1 + b x_2 + c) + \nu_i (a\tau_{\beta\beta}^{(1)} + b\tau_{\beta\beta}^{(2)} + c\tau_{\beta\beta}^{(3)}) - E_i \alpha_i T ,\end{aligned} \tag{4.9}$$

in the region Σ_i ($i=0, 1, 2, \ldots, m$).

From (2.1), (2.4), (4.4) it follows that $\sigma_{\alpha\beta}(x_1, x_2)$ must satisfy the equations

$$\sigma_{\alpha\beta,\beta} = 0 , \tag{4.10}$$

and the conditions

$$\sigma_{\alpha\beta} n_\beta = 0 , \text{ on } L_{m+1} , \; [\sigma_{\alpha\beta} n_\beta]_i = [\sigma_{\alpha\beta} n_\beta]_0 , \text{ on } L_i , \qquad (i=1, 2, \ldots, m \; ; \; \alpha, \beta = 1, 2) . \tag{4.11}$$

Obviously, the functions $v_\alpha(x_1, x_2)$ are the components of the displacement vector from the thermoelastic plane strain problem (4.8), (3.2), (4.10), (4.11) [4], [5], for temperature distribution $T=f(x_1, x_2)$. The conditions (2.5), (2.8) are satisfied on the basis of the relations (4.9). From (2.6) and (2.7) we obtain

$$\begin{aligned}a &= \frac{1}{d}(K_{22} m_1 - K_{21} m_2) , \quad b = \frac{1}{d}(K_{11} m_2 - K_{12} m_1) ,\\ c &= -a d_1 - b d_2 + p ,\end{aligned} \tag{4.12}$$

where

$$d_\beta = \frac{1}{d_3}\sum_{i=0}^{m}\int_{\Sigma_i}[E_i x_\beta + \nu_i \tau_{\rho\rho}^{(\beta)}]d\sigma\,,\quad d_3 = \sum_{i=0}^{m}\int_{\Sigma_i}[E_i + \nu_i \tau_{\rho\rho}^{(3)}]d\sigma\,,$$

$$p = \frac{1}{d_3}\sum_{i=0}^{m}\int_{\Sigma_i} F^{(i)} d\sigma\,,\quad F^{(i)} = \alpha_i E_i T - \nu_i \sigma_{\beta\beta}\,,$$

$$K_{\alpha\beta} = \sum_{i=0}^{m}\int_{\Sigma_i} h_\alpha^{(i)} x_\beta d\sigma\,,\quad h_\alpha^{(i)} = E_i x_\alpha + \nu_i \tau_{\beta\beta}^{(\alpha)} - d_\alpha[E_i + \nu_i \tau_{\beta\beta}^{(3)}]\,,$$

$$m_\beta = \sum_{i=0}^{m}\int_{\Sigma_i}[F^{(i)} - p(E_i + \nu_i \tau_{\alpha\alpha}^{(3)})]x_\beta d\sigma\,,\quad d = K_{11}K_{22} - K_{12}^2\,,\qquad (\alpha, \beta = 1, 2)\,. \quad (4.13)$$

Making use of the results from [3], we have $K_{12} = K_{21}$ and $d \neq 0$.

In what follows we seek to solve the problem (B).

We try to satisfy the conditions of the problem taking the displacements in the form (3.8), from the case of the homogeneous beams. Obviously, in our case the functions u_i^*, t_{ij}^* satisfy the conditions (4.1)–(4.3).

From (3.8) and (4.1) it follows that the function Φ must be continuous on Σ and that

$$[v_\alpha]_i - [v_\alpha]_0 = g_{\alpha i}\,,\ \text{on}\ L_i\,,\qquad (i = 1, 2, \ldots, m\ ;\ \alpha = 1, 2)\,, \quad (4.14)$$

where

$$g_{\alpha i} = [k_\alpha]_0 - [k_\alpha]_i\,,$$

$$[k_1]_i = -\nu_i x_1(\tfrac{1}{2}ax_1 + bx_2 + c) + \tfrac{1}{2}a\nu_i x_2^2\,,\ [k_2]_i = -\nu_i x_2(ax_1 + \tfrac{1}{2}bx_2 + c) + \tfrac{1}{2}b\nu_i x_1^2\,,$$
$$(i = 0, 1, 2, \ldots, m)\,. \quad (4.15)$$

The functions $\sigma_{\alpha\beta}$ given by (3.11) satisfy in each region Σ_i $(i = 0, 1, 2, \ldots, m)$ the equation (3.14). From (3.12) and (4.2), (4.3) we obtain the following conditions

$$[\sigma_{\alpha\beta}n_\beta]_i - [\sigma_{\alpha\beta}n_\beta]_0 = [h_\alpha]_i - [h_\alpha]_0\,,\ \text{on}\ L_i\,,\qquad (i = 1, 2, \ldots, m)\,,$$
$$\sigma_{\alpha\beta}n_\beta = [h_\alpha]_0\,,\ \text{on}\ L_{m+1}\,, \quad (4.16)$$

where

$$[h_\alpha]_i = -\frac{E_i \nu_i}{(1+\nu_i)(1-2\nu_i)} n_\alpha u_3^*(x_1, x_2, 0)\,,\qquad (i = 0, 1, 2, \ldots, m)\,. \quad (4.17)$$

The conditions for the existence of the solution of the problem (3.10), (3.14), (4.14), (4.16) are [4], [6]

$$-\sum_{j=0}^{m}\int_{\Sigma_j} F_\alpha d\sigma + \int_{L_{m+1}} h_\alpha ds + \sum_{j=1}^{m}\int_{L_j}([h_\alpha]_0 - [h_\alpha]_j)ds = 0\,,$$

$$-\sum_{j=0}^{m}\int_{\Sigma_j}(x_1 F_2 - x_2 F_1)d\sigma + \int_{L_{m+1}}(x_1 h_2 - x_2 h_1)ds +$$
$$+\sum_{j=1}^{m}\int_{L_j}\{[x_1 h_2 - x_2 h_1]_0 - [x_1 h_2 - x_2 h_1]_j\}ds = 0\,. \quad (4.18)$$

Taking into account the relations (3.15), (4.17), the divergence theorem and the fact that the normal vector is outward to Σ_0 the conditions (4.18) become

$$\sum_{j=0}^{m}\int_{\Sigma_j} t_{13}^*(x_1, x_2, 0)d\sigma = 0\,,\quad \sum_{j=0}^{m}\int_{\Sigma_j} t_{23}^*(x_1, x_2, 0)d\sigma = 0\,,$$

$$\sum_{j=0}^{m}\int_{\Sigma_j}[x_1 t_{23}^*(x_1, x_2, 0) - x_2 t_{13}^*(x_1, x_2, 0)]d\sigma = 0\,.$$

The above conditions are satisfied because the functions t^*_{ij} satisfy the conditions (2.5), (2.8).

The function $\Phi(x_1, x_2)$ must be continuous on Σ and satisfies in each region Σ_i ($i=0, 1, 2, \ldots, m$), the equation (3.21). From (3.12) and (4.2), (4.3) for $k=3$, we obtain the following conditions

$$\mu_0 \left[\frac{\partial \Phi}{\partial n}\right]_0 - \mu_i \left[\frac{\partial \Phi}{\partial n}\right]_i = q_i + \tau(\mu_0 - \mu_i)(x_2 n_1 - x_1 n_2), \text{ on } L_i,$$

$$(i = 1, 2, \ldots, m, \; m+1 \; ; \; \mu_{m+1} = 0), \quad (4.19)$$

where

$$q_i = (\mu_i - \mu_0) u^*_\alpha(x_1, x_2, 0) n_\alpha . \quad (4.20)$$

Let φ be the solution of the equation (3.23) in each region Σ_i, continuous on Σ and which satisfies the conditions

$$\mu_0 \left[\frac{\partial \varphi}{\partial n}\right]_0 - \mu_i \left[\frac{\partial \varphi}{\partial n}\right]_i = (\mu_0 - \mu_i)(x_2 n_1 - x_1 n_2), \text{ on } L_i,$$

$$(i = 1, 2, \ldots, m, \; m+1 \; ; \; \mu_{m+1} = 0). \quad (4.21)$$

The function $\varphi(x_1, x_2)$ exists [3]. If we make the substitution (3.25) where χ is a particular solution of the equation (3.21) in the region Σ_i ($i=0, 1, 2, \ldots, m$), it follows that the function $\psi(x_1, x_2)$ satisfies the equation (3.23) in each Σ_i, is continuous on Σ and satisfies the boundary conditions

$$\mu_0 \left[\frac{\partial \psi}{\partial n}\right]_0 - \mu_i \left[\frac{\partial \psi}{\partial n}\right]_i = p_i, \text{ on } L_i, \quad (i = 1, 2, \ldots, m, \; m+1 \; ; \; \mu_{m+1} = 0), \quad (4.22)$$

where

$$p_i = q_i - \mu_0 \left[\frac{\partial \chi}{\partial n}\right]_0 + \mu_i \left[\frac{\partial \chi}{\partial n}\right]_i . \quad (4.23)$$

The condition for the existence of the function ψ is [3]

$$\sum_{i=1}^{m+1} \int_{L_i} p_i ds = 0 .$$

Using the divergence theorem we get

$$\sum_{i=1}^{m+1} \int_{L_i} p_i ds = -\mu_0 \int_{L_{m+1}} u^*_\alpha(x_1, x_2, 0) n_\alpha ds - \sum_{i=0}^{m} \int_{\Sigma_i} \mu_i \Delta \chi d\sigma +$$

$$+ \sum_{i=1}^{m} \int_{L_i} (\mu_i - \mu_0) u^*_\alpha(x_1, x_2, 0) n_\alpha ds = \sum_{i=0}^{m} \int_{\Sigma_i} t^*_{33}(x_1, x_2, 0) d\sigma = 0 ,$$

because t^*_{33} satisfies the relation (2.6).

From (2.6)–(2.8) we can determine the constants a, b, c and τ. The conditions (2.5) are identically satisfied.

Thus we have solved the problem (B) and hence the initial problem is solved.

REFERENCES

[1] B. A. Boley and J. H. Weiner, *Theory of Thermal Stresses*, John Wiley, New York (1960).

[2] L. M. Milne–Thomson, *Antiplane Elastic Systems*, Springer (1962).

[3] N. I. Muskhelishvili, *Some Basic Problems of the Mathematical Theory of Elasticity*, Groningen (1953).

[4] D. I. Sherman, On the problem of plane strain in non-homogeneous media. *Non-homogeneity in Elasticity and Plasticity*, Pergamon Press, 3 (1959).

[5] D. Ieşan, Deformarea plana termoelastica a mediilor neomogene izotrope, *An. St. Univ. Iasi.*, Sect. I, 15 (1969) 463.

[6] D. I. Sherman, A statical plane problem in elasticity theory for non-homogeneous isotropic medium, *Papers of the Seysm. Inst. of the Acad. Sci. U.S.S.R.*, 86 (1938) (in Russian).